

Particle Resonances

with Application to Rings

A subset of this material is

- Overview presented by Reiser in § 3.8.6
pgs 166-171.
- Floquet Coordinates
- Perturbed Hells' equation in Floquet Coordinates
- Sources of and forms of Perturbation Terms.
- Unperturbed solution - relation to a simple harmonic oscillator.
- Perturbative analysis of perturbed Hells' equation and resonances
- Tune restrictions resulting from resonances and machine operating points.
- What will the effect of space-charge be in this picture.

Overview

In our treatment of single particle orbits of lattices with s -varying focusing we derived transverse particle orbit equations under the action of linear forces:

$$\begin{aligned} x''(s) + R_x(s) x(s) &= 0 \\ y''(s) + R_y(s) y(s) &= 0 \end{aligned}$$

Hill's Equations

Where

$$R_x(s), R_y(s)$$

describe the linear applied focusing fields of the lattice. These terms can also include linear space charge forces associated with the self-fields of a beam distribution. For certain classes of distributions (KV) the space-charge terms terminate at linear order.

In analyzing the Hill's equations we employed phase-amplitude methods:

$$x = A_x W_x(s) \cos \Psi_x(s)$$

$$W_x^2(s) \Psi_x'(s) = 1$$

This helped us identify the Courant-Snyder

invariant and helped us interpret the dynamics decoupled from the choice of initial conditions.

However, to make this formulation useful, one must solve for the amplitude function $W_x(s)$ or equivalently $\beta_x(s) = W_x^2(s)$. The equations for these amplitudes are nonlinear and in a periodic lattice we take them to be the periodic solution to the equation with the period of the lattice. These solutions are thought of as special functions of the lattice with respect to the case where space-charge is neglected. Because the solution of nonlinear equations is notoriously difficult, one might wonder if this formulation is useful beyond just identification of the existence of the Courant-Snyder invariant.

We will find that there are more uses of these methods beyond the identification of the Courant-Snyder invariant such as in simplifying the analysis of resonant particle instabilities in a periodic focusing lattice.

We will find that an extension of methods will allow us to map stable orbits described by Hill's equation with varying $R_x(s)$

$$\ddot{x}(s) + R_x(s)x(s) = 0$$

to a continuous oscillator:

$$\ddot{\tilde{x}}(s) + \frac{k_{po}^2}{\epsilon} \tilde{x}(s) = 0$$

$$\frac{k_{po}^2}{\epsilon} = \text{const.} > 0.$$

This will allow us to more simply analyze resonances in a perturbed oscillator where the perturbations (generally nonlinear) arise from imperfections in the focusing structure of the lattice and space-charge nonuniformities. The orbit equations with perturbations become:

$$\begin{aligned} \ddot{x}(s) + R_x(s)x(s) &= P_x(x, y, s, \delta, \dots) \\ \ddot{y}(s) + R_y(s)y(s) &= P_y(x, y, s, \delta, \dots) \end{aligned}$$

Perturbations Possible other
coupled variables

The formulation will also enable analytical treatment to aid understanding the full range of possibilities without laboriously exploring all possible conditions (various lattices, parameters, particle initial cond., etc.) numerically.

We will restrict analysis in this treatment to:

$$\gamma_b \beta_b = \text{const} \quad ; \quad \text{No Acceleration}$$

$$\delta = 0 \quad ; \quad \text{No momentum spread}$$

$$\phi = 0 \quad ; \quad \text{Neglect space-charge}$$

in order to more simply illustrate procedures.

Some comments on the effects of space-charge will be added at the end.

In our analysis we will also take the lattice to be periodic with

$$R_x(s + L_p) = R_x(s)$$

$$L_p = \text{Lattice Period}$$

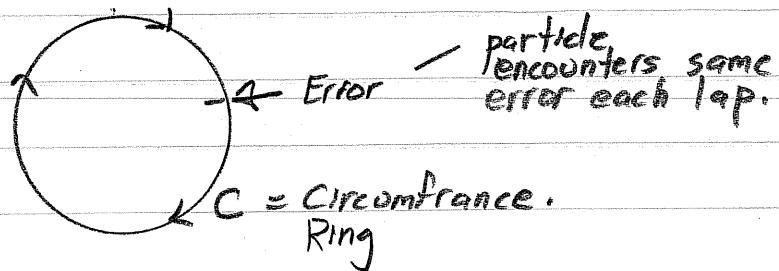
For a ring we also have the superperiodicity condition

$$P_x(s + C, \dots) = P_x(s + C, \dots)$$

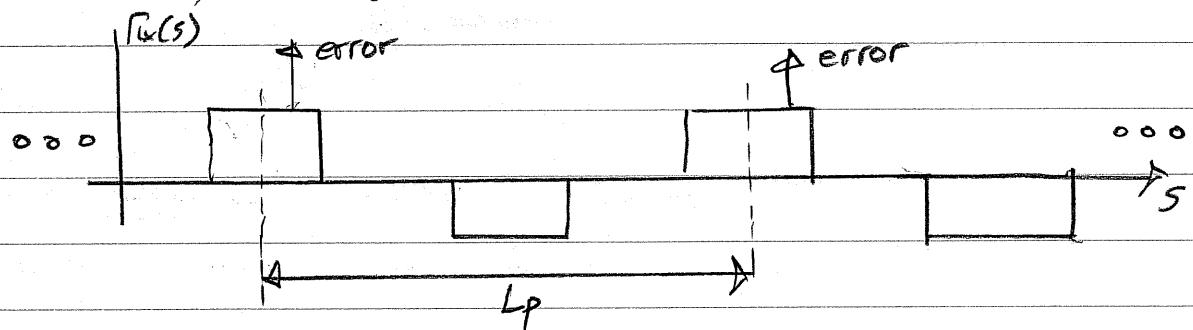
$$C = n \sqrt{L_p}$$

$$n\sqrt{} = \text{superperiodicity}$$

This applies to random errors in a ring



This error condition can also apply to systematic errors that occur each lattice period by taking $N = 1$.



This systematic error condition can apply to both ring and linear lattices.

In the following treatments, for simplicity we will often analyze only the x -orbit and drop subscript x 's etc.

Floquet Coordinates - Connection to a continuous oscillator

Denote: for a stable solution $x(s)$ to Hill's equation:

$$U = \frac{x(s)}{\sqrt{B(s)}} \quad B(s) = \beta\text{-fun. of lattice (periodic)}$$

$$\varphi = \frac{1}{2\pi} \int_{s_0}^s \frac{ds}{B(s)} = \frac{\Delta\psi(s)}{2\pi}$$

where the tune " ν_0 " is defined by:

We subscript by zero to denote the tune for zero self-field.

$$\nu_0 = \frac{\Delta\psi(NL_p)}{2\pi} ; \quad NL = \text{Superperiodicity}$$

ψ = phase of x -particle orbit

$$\Delta\psi(s) = \psi(s) - \psi(s_i)$$

ν_0 = number of undepressed particle oscillations in a ring.

For a linac we take $NL=1$ and ν_0 is the number of particle oscillations in a single period of the lattice.

Note that φ can be interpreted as a normalized angle (measured in particle phase advance) of the particle as it advances around a ring. That is, φ advances by 2π in one particle transit around the ring.

- In a linac φ advances by 2π through a lattice period.

Take φ to be the independent coordinate and

$$U = U(\varphi)$$

and transform the perturbed Hill's equation

$$x'' + P(s)x = p(x, y, s)$$

Here $p(x, y, s)$ stands for a perturbation that can represent nonlinear applied fields.

Let us assume

$$x = \sqrt{\beta'} U$$

$$x' = \frac{\beta' U}{2\sqrt{\beta'}} + \sqrt{\beta'} \frac{dU}{d\varphi} \frac{d\varphi}{ds}$$

Denote

$$\dot{U} = \frac{dU}{d\varphi}$$

and note that

$$\frac{d\varphi}{ds} = \frac{1}{2\sqrt{\beta}}$$

$$\Rightarrow x' = \frac{\beta' U}{2\sqrt{\beta}} + \frac{\dot{U}}{2\sqrt{\beta'}}$$

$$x'' = \frac{\beta'' U}{2\sqrt{\beta'}} - \frac{\beta'^2 U}{4\beta^{3/2}} - \cancel{\frac{\beta' \dot{U}}{2\sqrt{\beta}^{3/2}}} + \cancel{\frac{\beta' \ddot{U}}{2\sqrt{\beta}^{3/2}}} + \frac{\ddot{U}}{2\sqrt{\beta}^{3/2}}$$

Thus

$$x' = \frac{\beta' U}{2\sqrt{\beta'}} + \frac{\ddot{U}}{2\sqrt{\beta'}}$$

$$x'' = \frac{\beta'' U}{2\sqrt{\beta'}} - \frac{\beta'^2 U}{4\beta^{3/2}} + \frac{\ddot{\ddot{U}}}{2\sqrt{2}\beta^{3/2}}$$

and the perturbed Hill's equation becomes:

$$x'' + R x = P$$

$$\ddot{U} + \frac{\ddot{\ddot{U}}}{2} \left[\frac{\beta\beta''}{2} - \frac{\beta'^2}{4} + R\beta^2 \right] U = \frac{\ddot{\ddot{U}}}{2} \beta^{3/2} P$$

But

$$\frac{\beta\beta''}{2} - \frac{\beta'^2}{4} + R\beta^2 = 1$$

Betatron amplitude
by def. is the
periodic solution to
this equation.

by definition, and the perturbed Hill's equation
transforms to

$$\ddot{U} + \frac{\ddot{\ddot{U}}}{2} U = \frac{\ddot{\ddot{U}}}{2} \beta^{3/2} P(x, y, s)$$

$$U = U(\varphi) ; \quad \dot{U} = \frac{dU}{d\varphi}$$

$$\varphi = \int_{s_1}^s \frac{ds}{\sqrt{2\beta(s)}}$$

For $p=0$ (zero perturbation), the equation is just a simple harmonic oscillator equation with solution:

- Transform has mapped the time dependent unperturbed solution of Hill's equation to that of a simple harmonic oscillator.

$$U(\varphi) = U_0 \cos \omega_0 \varphi + \frac{\dot{U}_0}{\omega_0} \sin \omega_0 \varphi$$

where we take U_0 and \dot{U}_0 to be the particle initial conditions at $s=s_0$ with phase choice $\varphi=0$ at $s=s_0$.

The Floquet representation also simplifies the interpretation of the particle dynamics in response to perturbations due to the simple phase-space form of the unperturbed solution:

$$x = \sqrt{\epsilon'} \sqrt{\beta'} \cos \psi \quad ; \quad \psi = \omega_0 \varphi$$

$$U(\varphi) = \frac{x}{\sqrt{\beta'}} = \sqrt{\epsilon'} \cos \omega_0 \varphi$$

$$\dot{U}(\varphi) = \frac{dx}{d\varphi} = -\omega_0 \sqrt{\epsilon'} \sin \omega_0 \varphi$$

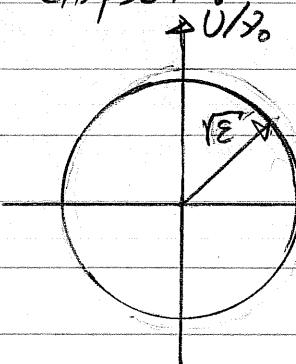
Thus the Courant-Snyder Invariant and the phase space structure are simply expressed in Floquet coordinates:

Courant-Snyder Invariant:

$$U^2 + \left(\frac{\dot{U}}{\omega_0} \right)^2 = \epsilon$$

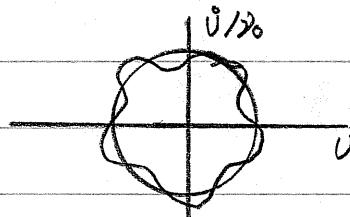
Equation of a circle rather than a rotated ellipse.

Phase Space ellipse:



Phase space is a circle.
for the unperturbed solution
to Hill's equation.

Perturbations are more easily understood as distortions
on circular phase-space!



The $U - U/I_0$ variables also preserve area
measures of phase space (a feature of the
transformation being symplectic).

* Jacobian

$$\int_{\text{ellipse}} dx^1 dx' = \int_{\text{ellipse}} du^1 du' |J| = \int_{\text{ellipse}} du^1 du \frac{1}{J}$$

// Proof:

$$x = \sqrt{\beta} U$$

$$x' = \frac{\beta'}{2\sqrt{\beta}} U + \sqrt{\beta} \frac{d\phi}{ds} \dot{U} = \frac{\beta'}{2\sqrt{\beta}} U + \frac{1}{2\sqrt{\beta}} \dot{U}$$

$$J = \det \begin{vmatrix} \frac{\partial x}{\partial U} & \frac{\partial x}{\partial \dot{U}} \\ \frac{\partial x'}{\partial U} & \frac{\partial x'}{\partial \dot{U}} \end{vmatrix} = \begin{vmatrix} \sqrt{\beta} & 0 \\ \frac{\beta'}{2\sqrt{\beta}} & \frac{1}{2\sqrt{\beta}} \end{vmatrix} = \frac{1}{2\beta}$$

$$dx^1 dx' = du^1 du \frac{1}{J}$$

//

We now analyze the effects of perturbations on the dynamics

$$\ddot{U} + \lambda_0^2 U = \lambda_0^2 \beta^{3/2} P(x, y, s)$$

*

Expand the perturbation in a power series

- Can always be done for all physical applied field perturbations.

$$P(x, y, s) = P_0(y, s) + P_1(y, s)x + P_2(y, s)x^2 + \dots$$

$$= \sum_{n=0}^{\infty} P_n(y, s) x^n$$

and take

$$x = \sqrt{\beta} U$$

to obtain

- * P represents a perturbation due to:
 - Systematic or random field errors in magnets etc.
 - Alignment error induced field terms etc.

$$\ddot{U} + \lambda_0^2 U = \lambda_0^2 \sum_{n=0}^{\infty} \beta^{\frac{n+3}{2}} P_n(y, s) U^n$$

To more simply illustrate resonances we will take the particle to move in the x -plane only ($y=0$ for all s). If this is not the case the formalism can be generalized by expanding the $P_n(y, s)$ in a power series in y and generalizing the notation for the Floquet coordinates to distinguish between the x - and y -planes, etc. The essential character of the more general analysis is illustrated by this simple case.

$$y(s) \equiv 0 \quad \text{to simplify picture}$$

In this special case ($y=0$) we expand each coefficient in the power series in a Fourier series as:

Here I implicitly assume a ring and keep φ as a 2π "phase" path variable in the ring for both systematic and random errors.

$$P_n(y=0, s) \beta^{\frac{n+3}{2}} = \sum_{k=-\infty}^{\infty} C_{n,k} e^{ikP\varphi}$$

$P = \begin{cases} 1 & - \text{A random perturbation (once in ring)} \\ \hbar \omega_0 & - \text{A periodic perturbation (every period)} \end{cases}$

$$C_{n,k} = \int_{-\pi/\omega_0}^{\pi/\omega_0} d\varphi P_n(y=0, s) \beta(s)^{\frac{n+3}{2}}$$

$$s = s(\varphi) ; \quad \varphi = \int_{s_0}^s \frac{ds}{\sqrt{2\beta(s)}}$$

Then the perturbed equation of motion becomes;

$$\ddot{U} + \omega_0^2 U = \omega_0^2 \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} C_{n,k} e^{ikP\varphi} U^n$$

For the case of small λ perturbations, this equation can be analyzed perturbatively to linear order as:

$$U = U_0 + \delta U ; \quad |U_0| \gg |\delta U|$$

where:

Simple Harmonic oscillator \rightarrow

$$\ddot{U}_0 + \omega_0^2 U_0 = 0$$

put unperturbed solution on this to leading order.

Simple \rightarrow
Harmonic oscillator
with driving terms.

$$\delta \ddot{U} + \omega_0^2 \delta U \cong \omega_0^2 \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} C_{n,k} e^{ikP\varphi} U_0^n$$

U_0 represents the unperturbed particle orbit.

The general solution to the equation for U_0 is:

can be expressed as:

$$U_0 = U_{0i} \cos(\omega_0 \varphi + \psi_i)$$

U_{0i}, ψ_i initial condition constants

Then,

$$\begin{aligned} U_0^n &= U_{0i} \left(\frac{e^{i(\omega_0 \varphi + \psi_i)} - e^{-i(\omega_0 \varphi + \psi_i)}}{2} \right)^n \\ &= \frac{U_{0i}}{2^n} \sum_{m=0}^n \binom{n}{m} e^{i(n-m)(\omega_0 \varphi + \psi_i)} e^{-im(\omega_0 \varphi + \psi_i)} \\ \binom{n}{m} &= \frac{n!}{m!(n-m)!} \quad \text{binomial coefficient.} \end{aligned}$$

$$= \frac{U_{0i}}{2^n} \sum_{m=0}^n \binom{n}{m} e^{i(n-m)\omega_0 \varphi} e^{i(n-m)\psi_i}$$

and the perturbed equation becomes:

$$\begin{aligned} \ddot{\delta U} + \omega_0^2 \delta U &\approx \omega_0^2 \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^n \binom{n}{m} C_{n,k} e^{i(n-m)\omega_0 \varphi} e^{i(n-m)\psi_i} \\ &\times e \end{aligned}$$

In general may take $\delta U = \delta U_h + \delta U_p$ where $\ddot{\delta U}_h + \omega_0^2 \delta U_h = 0$ but in this case we can take $\delta U_h = 0$ since this part of the perturbative solution is contained in U_0 . Thus only the particular solution need be found. From the properties of a driven harmonic oscillator, we know that no stable solution exists

whenever the frequency of the driving force equals that of the restoring force. Thus we have the resonance condition:

$$(n-2m)\nu_0 + pk = \pm \nu_0$$

$$n = 1, 2, 3, \dots ; m = 0, 1, 2, \dots n$$

$$k = -\infty, \dots 0, \dots \infty$$

$$P = \begin{cases} 1 & \text{A random perturbation (once per ring)} \\ N & \text{A periodic perturbation (every lattice period)} \end{cases}$$

For ν_0 satisfying this condition the perturbation will grow in amplitude. If the growth rate is sufficiently large, such tones will be unreliable operating points of the machine and the corresponding perturbation must be corrected. Since this is a linear analysis, the perturbations may be analyzed in turn:

Examples:

$n=0$ (Dipole Perturbation)

$n=0 \Rightarrow m=0$ and the resonance condition becomes:

$$\nu_0 = \pm pk \quad pk = \text{integer}$$

Therefore:

$$P = \begin{cases} 1 & \text{random pert} \\ N & \text{periodic perturbation in lattice} \end{cases}$$

$\nu_0 \neq \text{integer } (pk) \quad \text{for dipole } (n=0) \text{ perturbations}$

Note: For $P=N \Leftrightarrow$ form for periodic perturbation

$$\stackrel{\uparrow}{\text{error in }} \Rightarrow \nu_0 \neq Nk \quad k \text{ integer}$$

But: Always random - $\stackrel{\uparrow}{\text{every lattice period}}$ and disallowed tone

construction error terms ($P=1$)

In a ring?

values are few for large N .

$n=1$ Quadrupole Perturbation

The resonance conditions give:

$$\omega_0 + pk = \pm \omega_0$$

$$-\omega_0 + pk = \pm \omega_0$$

$\omega_0 + pk = \omega_0$ represents a special case that can be eliminated by "renormalizing" the driving force of the oscillator and $pk = 2\omega_0$ implies that:

$$pk = \begin{cases} 1; & \text{random pert. in ring} \\ 2\omega_0; & \text{periodic pert. in lattice} \end{cases}$$

$\omega_0 \neq \text{half-integer } (pk/2)$ for quadrupole ($n=1$) perturbations.

 $n=2$ Sextupole Perturbations

The resonance conditions give:

$$2\omega_0 + pk = \pm \omega_0$$

$$pk = \pm \omega_0$$

$$-2\omega_0 + pk = \pm \omega_0$$

These conditions are equivalent to:

$$pk = \begin{cases} 1; & \text{random pert. in ring} \\ 2\omega_0; & \text{periodic pert. in lattice} \end{cases}$$

$\omega_0 \neq \begin{cases} \text{integer } (pk) & \text{for sextupole} \\ \text{half-integer } (pk/2) & (n=3) \text{ perturbations} \\ \text{3rd-integer } (pk/3) \end{cases}$

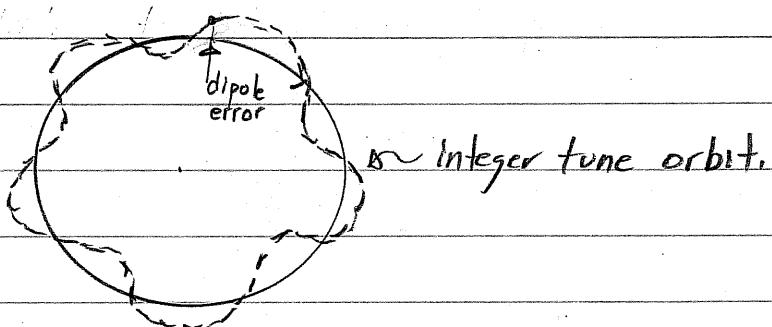
The integer and half-integer restrictions were already obtained with respect to the dipole and quadrupole cases. The 3rd integer is a new restriction.

Other cases similar

Aside Interpretation of low-order resonance conditions:

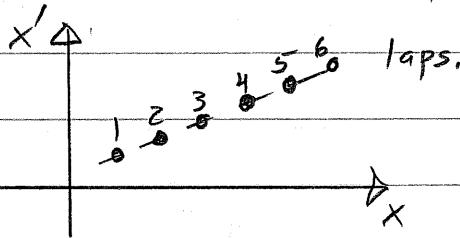
Dipole errors

Consider a ring with 1 dipole error along the azimuth of a ring:



on integer tune orbit.

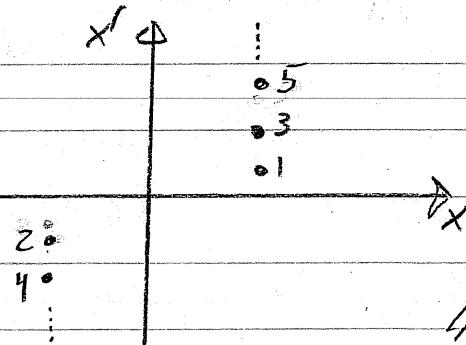
If the particle is oscillating with integer tune, then the particle will experience the error each time with the same phase^{of oscillation} and the particle trajectory will "walk-off" lap-to-lap in phase-space. Since the machine aperture is finite the particle will be lost.



Quadrupole Errors

For a single quadrupole error along the azimuth of a ring, a similar qualitative argument leads one to conclude that if the particle oscillates with $1/2$ integer tune that the orbit can "walk-off" lap-to-lap in phase space patterns as shown here:

as shown here:



The general resonance condition for x -plane motion can be summarized as:

$$M \omega_0 = N \quad M, N \text{ integers of the same sign.}$$

M = order resonance

Generally higher order numbers are less dangerous.

- Longer coherence path for validity of theory and coefficients generally smaller. Higher order can "wash" out.

In the general case particle motion is not restricted to the x -plane ($y \neq 0$) and a more general resonance analysis shows that:

$$M_x \omega_{0x} + M_y \omega_{0y} = N$$

ω_{0x} = x -plane tune

ω_{0y} = y -plane tune

M_x, M_y, N integers of the same sign.

Thus $|M_x| + |M_y|$ = order resonance.

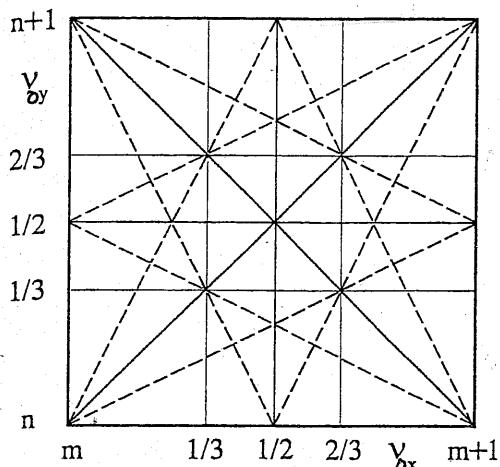
These restrictions are plotted order-by-order

in $\omega_x - \omega_y$ plots to find allowed tunes where the machine can be operated. Generally, lower order resonances are more dangerous since small effects can invalidate the ideal analysis and "wash out" higher order resonances.

Typical tune plots for up to 3rd order resonances!

$\mathcal{N}=1$ Superperiodicity

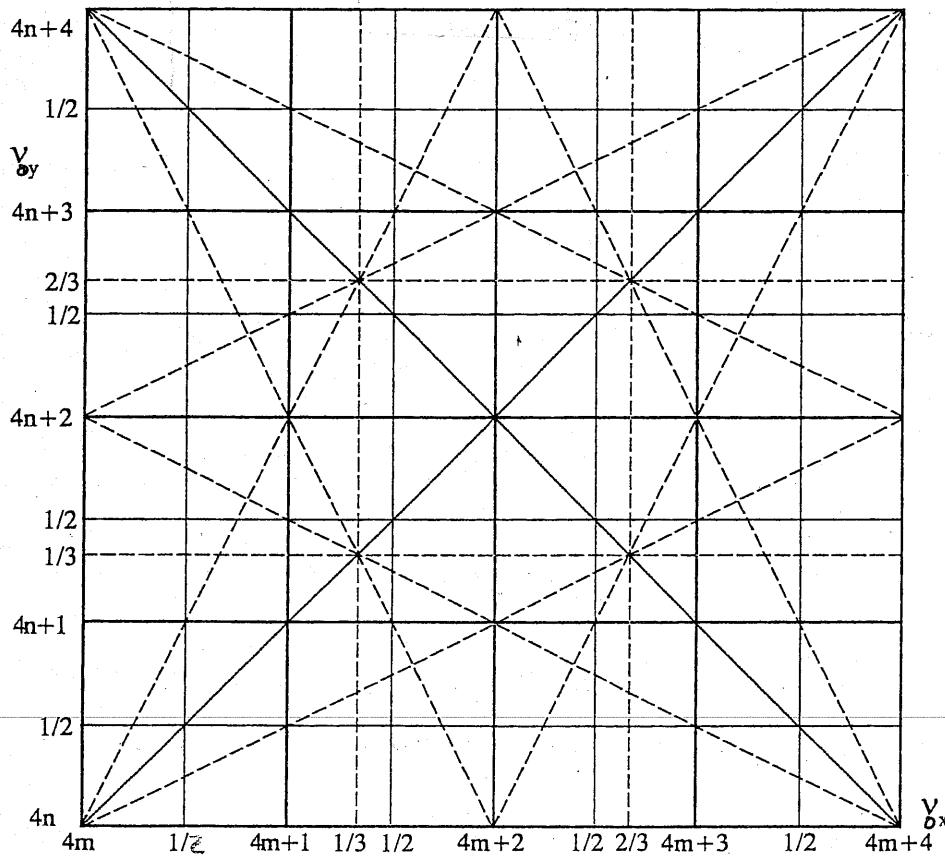
Machine operating point is chosen to avoid resonance lines.



Applicable to resonances
field errors from
random construction
errors,

From Wiedemann

$\mathcal{N}=4$ Superperiodicity



Applicable to
systematic
lattice
field errors
in a 4-period ring.

Note lesser
density of
resonance
lines.. than
In the $\mathcal{N}=1$
CASE.

Distinguishing between field errors from random construction ($p=1$) and systematic errors on the lattice ($p=N$) is important.

Random errors ($p=1$)

- Errors always present and can give low order resonances.
- Usually have weak amplitude coefficients and may be corrected.

Systematic errors ($p=N$)

- Lead to higher order resonances for large N and a lower density of resonance lines
 - Large symmetric rings (N large) have lesser restrictions from systematic errors
 - Practical issues such as construction cost and getting the beam into and out of the ring lead to smaller N
- Amplitude coefficients can be large and the systematic resonances can be strong and thus can be dangerous.

In practice, resonances higher than 3rd order need rarely be considered.

- Effects outside model tend to wash out higher order resonances.

The influence of space charge on resonances is an active area of research. (Very hard to calculate).

• Many beam rings experiments.

Effects of Space Charge on Resonances

Machine operating points are generally chosen far from low-order resonance lines. Coherent processes that shift tune values towards a low-order resonance are dangerous.

Coherent - same for each particle in distribution.

Incoherent - different (random) for each particle in distribution.

Tune shift limits are often called "Laslett Limits" because he ^{first} calculated such limits for many processes

- image charges
- image currents
- space charge

!

These restrictions are often taken to require $\Delta\nu \leq 1/4$ using coherent linear space-charge models. (KV model). However, since real space charge is more complicated, with strong incoherent waves and amplitude dependence, it is unclear how strict such a limit is for rings on long time scales ^(many ring laps) and simulations suggest surprisingly few troubles on short time scales (few to 10's of ring laps).

- Ongoing research topic!
- Univ of Maryland ^{experiment} may address soon.